IMEX Schemes for Advection-Diffusion Equations

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Outline

Motivation

- Explicit Scheme: Time step restriction due to stability conditions
- IMEX schemes: Weaker time step restriction
  (Implicit discretization of the diffusion part)

  $\Rightarrow$ Usage of greater time steps
  $\Rightarrow$ Saving computing time

Linear Scalar Advection-Diffusion Equation

$$\begin{aligned} u_t + \mathbf{a} \cdot \nabla u &= d \Delta u && \text{in } [0, T] \times \Omega, \\ u(t, \mathbf{x}) &= u_{\mathrm{B}}(t, \mathbf{x}) && \text{in } [0, T] \times \partial\Omega, \\ u(0, \mathbf{x}) &= u_0(\mathbf{x}) && \text{in } \Omega, \end{aligned}$$

$\mathbf{a}$: advection velocity,

d: diffusion parameter.

Stability Conditions of Explicit Schemes

For **explicit** time discretizations the following stability conditions have to be satisfied:

$$
\begin{aligned}
|\mathbf{a}| \frac{\Delta t}{\Delta x} &< 1 \\
d \frac{\Delta t}{\Delta x^2} &< \frac{1}{2}.
\end{aligned}
$$

Thus Δt has to be chosen so that

$$\Delta t < \min \left\{ \frac{\Delta x}{|\mathbf{a}|}, \frac{\Delta x^2}{2d} \right\}.$$

Stability Condition of IMEX Runge-Kutta Schemes

The idea of **IMEX** (Implicit-Explicit) schemes is to dispose of the diffusion stability condition

$$\cancel{d\frac{\Delta t}{\Delta x^2} < \frac{1}{2}}.$$

by discretizing the diffusion part $F^d(u) := d\Delta u$ *implicitly* and the advective part $F^a(u) := -\mathbf{a} \cdot \nabla u$ *explicitly*:

$$u_t = F^a(u) + F^d(u).$$

So the convection stability condition remains:

$$|\mathbf{a}|\frac{\Delta t}{\Delta x} < 1.$$

IMEX Algorithm

$$u_t = F^a(u) + F^d(u).$$

Explicit

$$F^a(u) := -\mathbf{a} \cdot \nabla u$$

Butcher Tables:

$\widehat{A}$, $\widehat{b}$ and $\widehat{c}$ of an explicit $(s+1)$-stage ERK

Implicit

$$F^d(u) := d\Delta u$$

Butcher Tables:

$$\widetilde{A} = \begin{bmatrix} 0 & 0 \\ * & A \end{bmatrix}, \widetilde{\mathbf{b}} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}$$

and

$$\widetilde{\mathbf{c}} = \begin{bmatrix} 0 \\ \mathbf{c} \end{bmatrix}.$$

A, b and c are the Butcher tables of a s-stage DIRK

IMEX Algorithm

Let u^n be given and u^{n+1} wanted.

$\widetilde{k}_1 = F^d(t^n, u^n)$

$\widehat{k}_1 = F^a(t^n, u^n)$

DO i=2,...σ

Solve for $\widetilde{k}_i$: $\widetilde{k}_i = F^d(t^n + c_i \Delta t, u_i)$

Evaluate $\widehat{k}_i$: $\widehat{k}_i = F^a(t^n + c_i \Delta t, u_i)$

$\Big\}$ with $u_i = u^n + \Delta t \left(\sum_{j=1}^{i} \widetilde{a}_{ij} \widetilde{k}_j + \sum_{j=1}^{i-1} \widehat{a}_{ij} \widehat{k}_j\right)$

END DO

$u^{n+1} = u^n + \Delta t \left(\sum_{j=1}^{\sigma} \widetilde{b}_j \widetilde{k}_j + \sum_{j=1}^{\sigma} \widehat{b}_j \widehat{k}_j\right)$

Implicit equation:

$$\tilde{k}_i = F^d\big(t^n + c_i \Delta t, u^n + \Delta t \left(\sum_{j=1}^{i} \tilde{a}_{ij} \tilde{k}_j + \sum_{j=1}^{i-1} \hat{a}_{ij} \hat{k}_j \right)\big)$$

Linear Equation System

Implicit equation:

$$\tilde{k}_i = F^d\big(t^n + c_i\Delta t, u^n + \Delta t\ \big(\sum_{j=1}^{i} \tilde{a}_{ij}\tilde{k}_j + \sum_{j=1}^{i-1} \hat{a}_{ij}\hat{k}_j\big)\big)$$

$$\Rightarrow \quad k_i - F^d(., dt\tilde{a}_{ii}\tilde{k}_i) = F^d\big(., u^n + \Delta t\ \big(\sum_{j=1}^{i-1} \tilde{a}_{ij}\tilde{k}_j + \sum_{j=1}^{i-1} \hat{a}_{ij}\hat{k}_j\big)\big)$$

$$\Rightarrow \quad \underbrace{(I - dt\tilde{a}_{ii}\mathbf{A}^d)}_{=:A}\underbrace{\tilde{\mathbf{k}}_\mathbf{i}}_{=:x} = \underbrace{\mathbf{A}^d\big(., \mathbf{u}^n + \Delta t\ \big(\sum_{j=1}^{i-1} \tilde{a}_{ij}\tilde{\mathbf{k}}_\mathbf{j} + \sum_{j=1}^{i-1} \hat{a}_{ij}\hat{\mathbf{k}}_\mathbf{j}\big)\big)}_{=:b}$$

$\rightarrow$ Solving the linear equation system with **GMRES**

Comparison of explicit and IMEX schemes

- To see the advantages of IMEX schemes, choose $\vec{a}$ and d so that

$$\frac{\Delta x^2}{2d} = \min\left\{\frac{\Delta x}{|\vec{a}|}, \frac{\Delta x^2}{2d}\right\}. \tag{1}$$

- Explicit time step is diffusion dominated:

$$\Delta t < \frac{\Delta x^2}{2d} < \frac{\Delta x}{|\vec{a}|}. \tag{2}$$

- Stability condition for IMEX:

$$\Delta t < \frac{\Delta x}{|\vec{a}|}.$$

Temporal schemes - Spatial Schemes

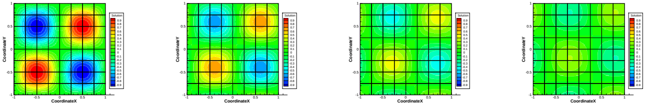
- **Temporal:**
  In the following we will consider schemes of 3^{rd} order:
  - 3-stage ERK
  - 3-stage IMEX (ERK + SDIRK): Ascher3
  - 4-stage IMEX (ERK + ESDIRK): Kennedy4
- **Spatial:**
  - Discontinuous Galerkin Spectral Element Method (DG-SEM), Code Flexi

Sinus Transport in 2D

- Exact solution of the linear scalar advection-diffusion equation in 2D

$$u(x, y, t) = \sin(\omega(x - a_1 - t)) \sin(\omega(y - a_2 - t)) e^{-2d\omega^2 t}.$$

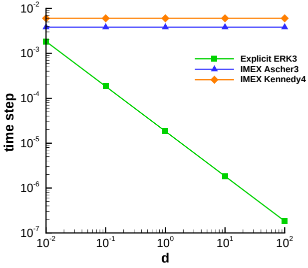
where $\vec{a} = (a_1, a_2) = (1, 1)$.

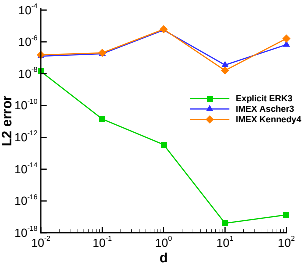


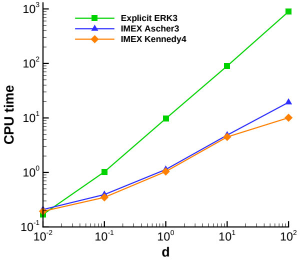
t=0 t=2 t=4 t=6

Parameter	Standard Values
Type of nodes	Gauss
Polynomial Order	8
Mesh size	8x8
ϵ_{GMRES} truncation	$\frac{10^{-4}}{d}$

Comparison of Explicit and IMEX schemes







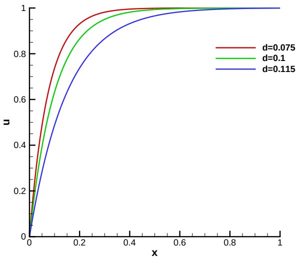
Singular Perturbation Problem in 1D

- Singular Perturbation Problem (SPP) can be seen as a model for the formation of a boundary layer
- Consider the linear scalar advection diffusion equation in 1D with a singular perturbation of the boundary values:

$$
\begin{aligned}
u_t + a\, u_x &= d\, u_{xx} && \text{in } [0, T] \times [0, 1], \\
u(0, t) &= 0 && \text{in } [0, T], \\
u(1, t) &= 1 && \text{in } [0, T], \\
u(x, 0) &= u_0(x) && \text{in } [0, 1].
\end{aligned}
$$

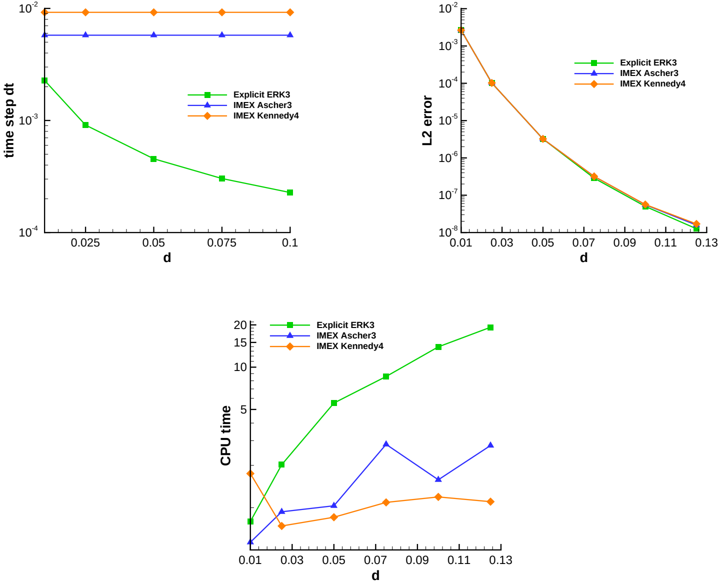
- Consider the stationary limit of the problem with $a = -1$.

$$\left| \|u_h(t^{n+1}) - u_{\text{ex}}\|_{L^2([0,1])} - \|u_h(t^n) - u_{\text{ex}}\|_{L^2([0,1])} \right| < c.$$



Parameter	Standard Values
Type of nodes	Gauss
Polynomial Order	8
Mesh size	8
ϵ_{GMRES} truncation	10^{-2}
Stationary truncation c	10^{-12}

Comparison of Explicit and IMEX schemes



Summary

- For the Sinus transport: Kennedy4 and Ascher3 are faster than the explicit scheme, but less accurate.
- For the convergence to the stationary limit: Kennedy4 is the fastest scheme
- Although Kennedy4 has 4 stages, it needs less or the same time than Ascher3 with the same L^2 accuracy for the considered test cases

Outlook

- Preconditioning of the implicit part
- Implementing IMEX schemes for Navier Stokes with a non linear solver (Newton)